New estimates for Weyl sums.

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1. Introduction.

By incorporating a new idea into a method based on one of Vinogradov, we showed in [**5**] that one can obtain substantially improved bounds for the size of smooth Weyl sums. In this note we investigate classical Weyl sums. From the historical point of view, improvements in estimates for classical Weyl sums have been rather thin on the ground. Over forty years ago, after a decade or so of intense effort, Hua [**2**] asserted that Vinogradov's method seemed to have reached a final stage. Indeed, it was only during the last few years that sufficient progress was made to provide a strong contradiction of Hua's assertion (see [**4**]), and even these improvements arose purely from superior mean value estimates. Thus, although the progress described below may seem numerically modest, its significance lies in the breaking of the old quasi-theoretical barriers (see Bombieri [**1**] for an interesting discussion of various approaches to the problem of estimating Weyl sums).

Let k be a natural number, and P be a large real number. For each k-tuple $\underline{\alpha} = (\alpha_1, \ldots, \alpha_k)$ of real numbers we define the *Weyl sum*, $f(\underline{\alpha}; P)$, by

$$f(\underline{\alpha}; P) = \sum_{1 \le x \le P} e(\alpha_k x^k + \ldots + \alpha_1 x),$$

where here, and throughout, we write $e(\alpha) = e^{2\pi i \alpha}$. The precise form of our results is to be found in section 3, the following upper bound being a simple corollary.

Theorem 1. *Let $\alpha_k \in \mathbb{R}$, and suppose that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$, $|\alpha_k - a/q| \le q^{-2}$ and $P \le q \le P^{k-1}$. Then*

$$f(\underline{\alpha}; P) \ll_{\varepsilon,k} P^{1-\rho(k)+\varepsilon},$$

where, when k is large, $\rho(k)^{-1} = \frac{3}{2}k^2 (\log k + O(\log\log k))$.

For comparison, Hua [**2**] obtains a similar result with the exponent satisfying $\rho(k)^{-1} = (4 + o(1))k^2 \log k$, this having been improved, by means of superior mean value estimates, by Wooley [**4**, Corollary 1.1] to the extent that "4" can be replaced by "2" in the latter conclusion.

Our estimate for $f(\underline{\alpha}; P)$ is based on an application of the large sieve inequality, the argument paralleling that of [**5**]. We let M be a real number with $P^{1/2} \le M \le P$ to

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be chosen later. Suppose that α_k satisfies the property that whenever $a \in \mathbb{Z}$, $q \in \mathbb{N}$ and $|\alpha_k - a/q| \le q^{-1}MP^{-k}$, one has $q > M$. By Dirichlet's Theorem, we may find $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a,q)=1$, $|\alpha_k - a/q| \le q^{-1}M^{-k}$ and $q \le M^k$. After obtaining an exponential sum based on well-factorable numbers by using Weyl shifts in section 2, we apply a Hölder's inequality argument to furnish ourselves in section 3 with a bound of the form

$$f(\underline{\alpha};P)^{2s} \ll M^{2s-1+\varepsilon}(P/M)^{\frac{1}{2}k(k-1)} \sum_{p\in\mathcal{P}_j} \sum_{h_1=1}^{s(P/M)} \cdots \sum_{h_{k-1}=1}^{s(P/M)^{k-1}} \left| \sum_{h_k=1}^{s(P/M)^k} b(\mathbf{h}) e(\alpha_k p^k h_k) \right|^2 .$$

Here, $b(\mathbf{h})$ denotes the number of solutions of the system of diophantine equations

$$\sum_{i=1}^{s} y_i^j = h_j \quad (1 \le j \le k)$$

with $1 \le y_i \le P/M$ $(1 \le i \le s)$, and the set $\mathcal{P}_j$ of primes satisfies the property that whenever $p_1, p_2 \in \mathcal{P}_j$ and $p_1^k \equiv p_2^k \pmod{q}$, then $p_1 \equiv p_2 \pmod{q}$. An analysis of the $\alpha_k p^k$ now shows that they are spaced at least $(2q)^{-1}$ apart modulo 1. Then by the large sieve inequality,

$$f(\underline{\alpha};P)^{2s} \ll M^{2s-1+\varepsilon}(P/M)^{\frac{1}{2}k(k-1)} \left(q + s(P/M)^k\right) \sum_{\mathbf{h}} b(\mathbf{h})^2,$$

the last sum being estimated by using a suitable version of Vinogradov's mean value theorem. We are able to deal with the possibility that q is large by using a complementary bound on $f(\underline{\alpha};P)$, useful only in the latter situation. Thus we may take $M = P^\lambda$ with a suitable $\lambda > \frac{1}{2}$, and discard those q with $q > P^{k(1-\lambda)}$. Thereby we achieve the desired bounds on $f(\underline{\alpha};P)$.

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2. Preliminary observations.

We start by recalling Vinogradov's mean value theorem, which we describe here in the strongest form currently available. Let s and k be natural numbers, and let P be a large real number (in terms of s and k). Also, let ε be a positive number sufficiently small in terms of s and k. We use $\ll$ and $\gg$ to denote Vinogradov's well-known notation, implicit constants depending at most on s, k and ε. Further, we write $\|x\|$ for $\min_{y\in\mathbb{Z}}|x-y|$. For the sake of conciseness, we adopt the convention that whenever ε appears in a statement, then for each $\varepsilon>0$ the statement holds. Thus the "value" of ε may change from statement to statement.

We define $J_{s,k}(P)$ to be the number of solutions of the system of diophantine equations

$$\sum_{i=1}^{s}(x_i^j-y_i^j)=0\quad(1\le j\le k),$$

with $1\le x_i,y_i\le P$ $(1\le i\le s)$. Thus

$$J_{s,k}(P)=\int_{[0,1)^k}|f(\underline{\alpha};P)|^{2s}d\underline{\alpha}.$$

We shall say that an exponent $\Delta_{s,k}$ is *permissible* whenever the exponent has the property that $J_{s,k}(P)\ll P^{\lambda_{s,k}}$, with $\lambda_{s,k}=2s-\frac{1}{2}k(k+1)+\Delta_{s,k}$. It follows easily that any permissible exponent $\Delta_{s,k}$ is non-negative, and moreover, without loss of generality, that $\Delta_{s,k}\le\frac{1}{2}k(k+1)$. The following lemma provides us with permissible exponents when k is large.

Lemma 1. *There exists an absolute constant k_0 such that whenever $k\ge k_0$ and*

$$1\le r\le[k(\log k-\log\log k)]+1,$$

then the exponent

$$\Delta_{rk,k}=k^2\log k\left(1-\tfrac{2}{k}(1-1/\log k)\right)^r.$$

is permissible.

Proof. This is [**4**, Theorem 1.2].

We now provide a lemma which enables us, from our original Weyl sum, to generate exponential sums of a suitable form for our later arguments. This we achieve through the use of Weyl shifts. A comparison with a modern treatment of the problem (see, for example, [**3**, equation (5.23)]) will reveal that we have reversed the rôles of the variables. For the sake of conciseness, for each real k-tuple $\underline{\alpha}=(\alpha_1,\dots,\alpha_k)$, we write

$$\psi(x;\underline{\alpha})=\alpha_kx^k+\ldots+\alpha_1x,$$

so that

$$f(\underline{\alpha};P)=\sum_{1\le x\le P}e(\psi(x;\underline{\alpha})).$$

Lemma 2. *Suppose that* $\mathcal{A} \subseteq [1, P] \cap \mathbb{Z}$ *satisfies card*$(\mathcal{A}) = Q$. *Then*

$$f(\underline{\alpha}; P) \ll Q^{-1} \log 2P \sum_{1 \le x \le 2P} \sup_{\beta \in [0,1)} |g(\underline{\alpha}, \beta; x)|,$$

where

$$g(\underline{\alpha}, \beta; x) = \sum_{y \in \mathcal{A}} e(\psi(x-y; \underline{\alpha}) + \beta y).$$

Proof. For each $y \in \mathcal{A}$, we have

$$f(\underline{\alpha}; P) = \sum_{1+y \le x \le P+y} e(\psi(x-y; \underline{\alpha})).$$

Thus

$$\begin{aligned} Qf(\underline{\alpha}; P) &= \sum_{y \in \mathcal{A}} \sum_{1+y \le x \le P+y} e(\psi(x-y; \underline{\alpha})) \\ &= \sum_{1 \le x \le 2P} \sum_{\substack{y \in \mathcal{A} \\ x-P \le y \le x-1}} e(\psi(x-y; \underline{\alpha})). \end{aligned} \tag{1}$$

But

$$\begin{aligned} \sum_{\substack{y \in \mathcal{A} \\ x-P \le y \le x-1}} e(\psi(x-y; \underline{\alpha})) &= \int_0^1 \sum_{y \in \mathcal{A}} e(\psi(x-y; \underline{\alpha}) + \beta y) \sum_{x-P \le u \le x-1} e(-u\beta) d\beta \\ &\ll \int_0^1 |g(\underline{\alpha}, \beta; x)| \min\left\{P, \|\beta\|^{-1}\right\} d\beta \\ &\ll \log 2P \sup_{\beta \in [0,1)} |g(\underline{\alpha}, \beta; x)|. \end{aligned} \tag{2}$$

The lemma now follows on combining (1) and (2).

Finally, we conclude with an estimate for Weyl sums of use for large moduli.

Lemma 3. *Let* $\alpha_k \in \mathbb{R}$, *and suppose that there exist* $a \in \mathbb{Z}$ *and* $q \in \mathbb{N}$ *with* $(a, q) = 1$ *and* $|\alpha_k - a/q| \le q^{-2}$. *Suppose also that* r *is an integer with* $P^r \le q \le P^{k-r}$. *Then for* $s \ge \frac{1}{2}k(k-1)$, *and any permissible exponent* $\Delta_{s,k-1}$, *we have the bound*

$$f(\underline{\alpha}; P) \ll P^{1-\mu(k)+\varepsilon},$$

where

$$\mu(k) = \frac{r - \Delta_{s,k-1}}{2rs}.$$

Proof. This is Bombieri [**1**, Corollary 1 to Theorem 8].

3. Upper bounds for small moduli.

When α_k is close to a rational a/q with q small, we use the large sieve inequality to provide an upper bound for $f(\underline{\alpha};P)$.

Lemma 4. *Let M be a real number satisfying $1 \le M \le P$. Let $\alpha_k \in \mathbb{R}$, and suppose that $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a,q)=1$, $|q\alpha_k - a| \le M^{-k}$ and either $|q\alpha_k - a| > MP^{-k}$ or $q > M$. Then for each $s \in \mathbb{N}$, and any permissible exponent $\Delta_{s,k}$,*

$$f(\underline{\alpha};P) \ll q^{\varepsilon} P^{1+\varepsilon} \left(M^{-1}(P/M)^{\Delta_{s,k}} \left(1 + q(P/M)^{-k}\right)\right)^{1/2s}.$$

Proof. We take $\mathcal{P}$ to be the set of primes p satisfying $2^{-\frac{1}{k-1}}M < p \le 2^{-\frac{1}{k}}M$, and apply Lemma 2 with

$$\mathcal{A} = \{n \in \mathbb{Z} \cap [1,P] : n = py,\ p \in \mathcal{P},\ 1 \le y \le P/M\}$$

to establish that

$$f(\underline{\alpha};P) \ll P^{\varepsilon - 1} \sum_{1 \le x \le 2P} \sup_{\beta \in [0,1)} \left| \sum_{w \in \mathcal{A}} e(\psi(x-w;\underline{\alpha}) + \beta w) \right|.$$

On expanding $\psi(x-w;\underline{\alpha})$ as a polynomial in w, we deduce that there exist real numbers $\gamma_1, \ldots, \gamma_{k-1}$ such that

$$f(\underline{\alpha};P) \ll P^{\varepsilon} \left| \sum_{w \in \mathcal{A}} e(\alpha_k w^k + \gamma_{k-1} w^{k-1} + \ldots + \gamma_1 w) \right|.$$

Thus an application of Hölder's inequality gives

$$f(\underline{\alpha};P)^{2s} \ll P^{\varepsilon} M^{2s-1} \sum_{p \in \mathcal{P}} \left| \sum_{1 \le y \le P/M} e(\alpha_k (py)^k + \gamma_{k-1}(py)^{k-1} + \ldots + \gamma_1 py) \right|^{2s}. \quad (3)$$

When $(h,q)=1$, the number of solutions of the congruence $x^k \equiv h \pmod{q}$ with $1 \le x \le q$ is $O(q^{\varepsilon})$. Hence there is an $L \ll q^{\varepsilon}$ such that the primes $p \in \mathcal{P}$ can be divided into L classes $\mathcal{P}_1, \ldots, \mathcal{P}_L$, such that for any two distinct elements p_1, p_2 in a given class $\mathcal{P}_j$, we have $p_1^k \equiv p_2^k \pmod q$ if and only if $p_1 \equiv p_2 \pmod q$. Then on writing $b(\mathbf{h})$ for the number of solutions of the system of diophantine equations

$$\sum_{i=1}^{s} y_i^j = h_j \quad (1 \le j \le k)$$

with $1 \le y_i \le P/M$ $(1 \le i \le s)$, we deduce from (3) that there exists a j with $1 \le j \le L$ such that

$$f(\underline{\alpha}; P)^{2s} \ll (qP)^{\varepsilon} M^{2s-1} \sum_{p \in \mathcal{P}_j} \left| \sum_{h_1=1}^{H_1} \ldots \sum_{h_k=1}^{H_k} b(\mathbf{h}) e(\alpha_k h_k p^k + \phi(\underline{\gamma}, \mathbf{h}, p)) \right|^2 ,$$

where $H_i = s(P/M)^i$ $(1 \le i \le k)$, and $\phi(\underline{\gamma}, \mathbf{h}, p) = \gamma_{k-1} h_{k-1} p^{k-1} + \ldots + \gamma_1 h_1 p$. Thus, on applying Cauchy's inequality,

$$f(\underline{\alpha}; P)^{2s} \ll (qP)^{\varepsilon} M^{2s-1} (P/M)^{\frac{1}{2}k(k-1)} \sum_{p \in \mathcal{P}_j} \sum_{h_1=1}^{H_1} \ldots \sum_{h_{k-1}=1}^{H_{k-1}} \left| \sum_{h_k=1}^{H_k} b(\mathbf{h}) e(\alpha_k p^k h_k) \right|^2 .$$

If $p_1, p_2 \in \mathcal{P}_j$ and $p_1 \not\equiv p_2 \pmod q$, then we have $p_1^k \not\equiv p_2^k \pmod q$. Then on recalling that $|q\alpha_k - a| \le M^{-k}$, we deduce that

$$\|\alpha_k(p_1^k - p_2^k)\| \ge \left\| \frac{a}{q}(p_1^k - p_2^k) \right\| - \tfrac{1}{2}q^{-1} \ge \tfrac{1}{2}q^{-1}. \tag{4}$$

We divide into cases.

(i) Suppose that $q > M$. Then the elements of $\mathcal{P}_j$ are distinct mod q, and it follows from (4) that for $p \in \mathcal{P}_j$, the $\alpha_k p^k$ are spaced at least $\frac{1}{2}q^{-1}$ apart modulo 1.

(ii) Suppose that $q \le M$. Then by hypothesis we have $|q\alpha_k - a| > MP^{-k}$. Given any two distinct elements p_1, p_2 of $\mathcal{P}_j$ with $p_1 \not\equiv p_2 \pmod q$, we may conclude as in case (i) that αp_1^k and αp_2^k are spaced at least $\frac{1}{2}q^{-1}$ apart modulo 1. Thus we are left to consider the situation in which $p_1 \equiv p_2 \pmod q$, but $p_1 \ne p_2$, in which case

$$\|\alpha_k(p_1^k - p_2^k)\| = \|(\alpha_k - a/q)(p_1^k - p_2^k)\| = |\alpha_k - a/q| \cdot |p_1^k - p_2^k|.$$

Since $p_1 - p_2$ is a non-zero multiple of q, and $p_1 > 2^{-\frac{1}{k-1}} M$ and $p_2 > 2^{-\frac{1}{k-1}} M$, we have

$$\|\alpha_k(p_1^k - p_2^k)\| \ge |\alpha_k - a/q| (2^{-\frac{1}{k-1}} M)^{k-1} q > \tfrac{1}{2}(P/M)^{-k}.$$

Therefore, in this case, for $p \in \mathcal{P}_j$ the αp^k are spaced at least $\frac{1}{2}\min\{q^{-1}, (P/M)^{-k}\}$ apart modulo 1.

Then in either case, by the large sieve inequality (see, for example, [**3**, Lemma 5.3]) we have

$$\sum_{p\in\mathcal{P}_j}\left|\sum_{h_k=1}^{H_k} b(\mathbf{h})e(\alpha_k p^k h_k)\right|^2 \ll \left(q+(P/M)^k\right)\sum_{h_k=1}^{H_k}|b(\mathbf{h})|^2.$$

Thus

$$\begin{aligned} f(\underline{\alpha};P)^{2s} &\ll (qP)^{\varepsilon}M^{2s-1}(P/M)^{\frac{1}{2}k(k-1)}\left(\sum_{\mathbf{h}} b(\mathbf{h})^2\right)\left(q+(P/M)^k\right)\\ &\ll (qP)^{\varepsilon}M^{2s-1}(P/M)^{\frac{1}{2}k(k+1)}\left(1+q(P/M)^{-k}\right)J_{s,k}(P/M)\\ &\ll q^{\varepsilon}P^{2s+\varepsilon}M^{-1}(P/M)^{\Delta_{s,k}}\left(1+q(P/M)^{-k}\right). \end{aligned}$$

The lemma now follows immediately.

Theorem 2. *Let r be an integer with $1\le r\le \frac{1}{2}k$, and write $\lambda = 1-\frac{r}{k}$. Let $\alpha_k\in\mathbb{R}$, and suppose that whenever $a\in\mathbb{Z}$ and $q\in\mathbb{N}$ satisfy $(a,q)=1$ and $|\alpha_k - a/q|\le q^{-1}P^{\lambda-k}$, then one has $q>P^{\lambda}$. Then if s and t are positive integers with $s\ge \frac{1}{2}k(k-1)$, and the exponents $\Delta_{s,k-1}$ and $\Delta_{t,k}$ are permissible, we have the bound*

$$f(\underline{\alpha};P)\ll P^{\varepsilon}\left(P^{1-\mu(k)}+P^{1-\nu(k)}\right),$$

where

$$\mu(k)=\frac{r-\Delta_{s,k-1}}{2rs}$$

and

$$\nu(k)=\frac{k-r(1+\Delta_{t,k})}{2tk}.$$

Proof. Write $M=P^{\lambda}$. By Dirichlet's Theorem, there exist $a\in\mathbb{Z}$ and $q\in\mathbb{N}$ with $(a,q)=1$, $q\le M^k=P^{k-r}$ and $|\alpha_k-a/q|\le q^{-1}M^{-k}$. If $q>(P/M)^k=P^r$, then we may apply Lemma 3 to obtain the bound

$$f(\underline{\alpha};P)\ll P^{1-\mu(k)+\varepsilon}.$$

Thus we may assume that $q\le (P/M)^k$. In this second case we apply Lemma 4 to establish the estimate

$$f(\underline{\alpha};P)\ll P^{1+\varepsilon}\left(P^{-\lambda+(1-\lambda)\Delta_{t,k}}\right)^{1/2t} = P^{1-\nu(k)+\varepsilon}.$$

This completes the proof of the theorem.

We now explore the consequences of our new estimate when k is large.

Corollary 1. *Let $\mathfrak{m}_\lambda$ denote the set of $\alpha \in \mathbb{R}$ such that whenever $a \in \mathbb{Z}$, $q \in \mathbb{N}$, $(a, q) = 1$ and $|q\alpha - a| \le P^{\lambda - k}$, one has $q > P^\lambda$. Then there is a natural number $k_0(\varepsilon)$ with the following property. When $k \ge k_0(\varepsilon)$, there are real numbers $\lambda = \lambda(k)$ and $\sigma(k)$ with*

$$k^{-1/2} \ll 1 - \lambda \ll k^{-1/2},$$

and

$$\sigma(k)^{-1} = \tfrac{3}{2}k^2(\log k + O(\log\log k)),$$

and such that

$$\sup_{\alpha_k \in \mathfrak{m}_\lambda} |f(\underline{\alpha}; P)| \ll P^{1-\sigma(k)+\varepsilon}.$$

Proof. We put

$$t = k\left[\tfrac{3}{4}k(\log k + 2\log\log k)\right],$$

and

$$s = (k-1)\left[\tfrac{3}{4}(k-1)(\log(k-1) + 2\log\log(k-1))\right].$$

Then by Lemma 1, there are permissible exponents $\Delta_{t,k}$ and $\Delta_{s,k-1}$ with

$$\begin{aligned} \Delta_{t,k} &\le k^2 \log k \exp\left(-\tfrac{3}{2}(\log k + 2\log\log k + O(1))\right) \\ &\ll k^{1/2}(\log k)^{-2}, \end{aligned}$$

and similarly

$$\Delta_{s,k-1} \ll k^{1/2}(\log k)^{-2}.$$

Thus, on taking $r = [k^{1/2}]$, in the notation of Theorem 2 we obtain

$$\begin{aligned} \mu(k) &\ge \frac{k^{1/2} + O(k^{1/2}(\log k)^{-2})}{2k^{1/2}s} \\ &\ge \left(\tfrac{3}{2}k^2\left(\log k + O(\log\log k)\right)\right)^{-1}. \end{aligned}$$

Also,

$$\begin{aligned} \nu(k) &\ge \frac{k + O(k(\log k)^{-2})}{2tk} \\ &\ge \left(\tfrac{3}{2}k^2\left(\log k + O(\log\log k)\right)\right)^{-1}. \end{aligned}$$

The lemma therefore follows immediately from Theorem 2.

Theorem 1 follows almost immediately from the corollary. For suppose that α_k satisfies the hypotheses of Theorem 1. We take λ to be as in the statement of the corollary. Suppose that $b \in \mathbb{Z}$ and $r \in \mathbb{N}$ satisfy $(b, r) = 1$ and $|r\alpha - b| \leq P^{\lambda-k}$. Then either $q = r$, or else

$$\frac{1}{qr} \leq \left| \frac{a}{q} - \frac{b}{r} \right| \leq q^{-2} + r^{-1} P^{\lambda-k}.$$

In the latter case, we have $q(1 - qP^{\lambda-k}) \leq r$, so that since $q \leq P^{k-1}$, we have $q \leq 2r$. Then in either case, $r \geq \frac{1}{2}q > P^{\lambda}$. Thus $\alpha_k \in \mathfrak{m}_\lambda$, and the desired conclusion follows from the corollary.

We note that the conclusion of Theorem 2 may be improved a little for smaller k by making use of a refined version of Lemma 3, which exploits the size of q more effectively. However, at this point we choose not to consider the consequences for smaller k of the above methods.

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